#### **3. Predetor–pray model. Phase space analysis**

**3.1. Introduction**

**Object** Coexistence of two biological species

**Classification** There exists three non-trivial form of the coexistence: ++, +-, --

**Object clarification** coexistence +-, that is the the predator–pray system

**Foundation** System of differential equations, phase space

**Aim** Definition of the law of change in the number of both species depending on the conditions of the process

**3.2. Predetor–pray model**

**General supposition:** The prey has a positive effect on the predator.

 The predator has a negative influence on the prey

 The function *x*1*=x*1(*t*) describes the number of preys at the time *t*.

 The function *x*2*=x*2(*t*) describes the number of predators at the time *t*.

Consider now a coexistence of two species when the first of them is a food for the second. If only the first species (prey) lived in this environment, then it would have a natural increase of *ε*1, which is constant and positive if is assumed that the prey do not lack food. Then, in the absence of predators, an exponential increase in the abundance of prey would be observed, as was the case in the previously considered model of evolution of the species with an unlimited amount of food. If the second species (predators) exists in isolation, then due to lack of food (i.e., prey), it would have a negative population growth of –*ε*2 that is the coefficient of extinction of predators in the absence of prey. The outcome here would be the complete extinction of predators.

These species have serious influence on each other under their coexistence in a limited area. Obviously, the increase in the abundance of prey should decrease, and the more, the higher the abundance of predators. This is due to the fact that a greater abundance of predators need an appropriate amount of food, i.e. the prey. On the other hand, the growth of predators should increase the stronger, the higher the prey abundance, since in these conditions a greater predators abundance will be provided with nutritious food. As a result, we get

  (1)

where *γ*1 and *γ*2 are positive proportionality coefficients. We have the ***system of differential equations***. There are the ***Volterra – Lotka equations***.

For obtaining its uniqie solution, it is necessary to add the initial conditions. Suppose we now the initial numbers of species *x*10 and *x*20. Now we have theinitial conditions

 *xi*(0) = *xi*0, *i=*1,2. (2)

The Caushy problem (1), (2) id called the ***predator–prey model***. This model has six parameters that are four coefficients of the equations (1) and two initial states.

**3.3. Phase plane and phase curve**

Our system id described by two function *x*1*=x*1(*t*) and *x*2*=x*2(*t*). If we fixed the concrete time moment *t*, then the state of the system id described by two numbers *x*1(*t*) and *x*2(*t*), which are the values of the considered functions at this time. We can interpreted it as a coordinate of a point
(*x*1(*t*),*x*2(*t*)) on the plane with axes *x*1 and *x*2. This plane is called the ***phase plane*** of the system of the considered two differential equations.

If the time changes, then the position of this point changes too. If we consider a time interval from the initial time *t=*0 to a final time, we get a curve that is called the ***phase curve***. The property of the phase curve gives us an information about the behavior of the system. Of course, it depends from all system parameters.

 **3.4. Equilibrium state for the system of two differential equations**

We considered an equilibrium position of a system described by a unique differential equation. This is a solution of the considered equation, which does not depend on time. Now we have the system of two defferential equations

 , (3)

where the functions *f*1 and *f*2 are given. The system can be in the equilibrium position if both functions *x*1 and *x*2 do not change. Hence, the derivatives of these functions are zero. From the system (3) for this case, it follows –

  (4)

Now we have two algebraic equations with respect to two unknow values *x*1 and *x*2. Therefore, for finding equilibrium positions of the system of differential equations (3), it is necessary to solve the system of algebraic equations (4).

**3.5. Equilibrium states for the predator–preysystem**

The considered system (1) is the partial case of the general system (3). Determine the corresponding equations (4):

 (*ε*1–*γ*1*x*2)*x*1 = 0, (5)

 (*γ*2*x*1–*ε*2)*x*2 = 0. (6)

The equality (5) can be true for cases, i.e. for *x*1=0 or for *x*2=*ε*1/*γ*1. Suppose *x*1=0. Then the equality (6) takes the form *ε*2*x*2=0, so *x*2=0. Suppose now *x*2=*ε*1/*γ*1. In this case, the equality (6) can be true only for *x*1=*ε*2/*γ*2.

**Conclusion**:predator–preymodel has two equilibrium positions. There are trivial state *x*1=0, *x*2=0 and non-trivial state *x*1=*ε*2/*γ*2, *x*2=*ε*1/*γ*1.

The trivial equilibrium is clear. We do not have any population. Therefore, the system is in this state, i.e. the numbers of both specias are equal to zero for any time. The sense of the non-trivial equilibrium position requires an additional analysis.

**3.6. Trivial solutions of the predator–preysystem**

The first trivial solution is the system (1) with zero intial conditions, which corresponds to the absence of both species and trivial equivilibrium position. However, we can consider additional trivial solutions of this system with presence only unique species.

Suppose the predators are absent, i.e., *x*20=0. Therefore, the second species is in the equilibrium position, so we will not have any predator for all times. Now the first equality (1) takes the form



This is the Malthus equation with positive value of the growth coefficient. We know that the function *x*1 increases unboundedly.

Suppose now the preys are absent, i.e., *x*10=0. Therefore, the first species is in the equilibrium position, so we will not have any prey for all times. Now the first equality (1) takes the form

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This is the Malthus equation with negative growth coefficient. We know that the function *x*2 decreases to zero.

**Conclusion**. The prey population unboundedly increses if the predators are absent.

The predator population dies out if the preys are absent.

**3.7. Analysis of the system of differential equation on the phase plane**

Suppose now the solution of the system (3) at the arbitrary time *t.* The corresponding pair of values *x*1(*t*) and *x*2(*t*) gives us a point *M* of the phase plane; see Figure 3.1. Put it to the system (3). Consider first of them, i.e.,



The function *f*1 is given, and the values *x*1(*t*) and *x*2(*t*) are known. Therefore, we can check the sign of the derivative  If this valueis positive, the function *x*1 increases at this point. This function decreases there if this derivative here is negative. We can use analogical method for the function *x*2. For example, if both derivatives are positive at the point *t*, then both functions increase there; see Figure 1.



Figure 1. Change direction of the functions *x*1 and *x*2for the abstract system.

**Conclusion**. For analysis of the system of differential equations, it is necessary to divide the phase plane by parths such that for any part, the derivatives of the considered functions have the concrete sign: both positive, both negative, first positive and second negative, and first negative and second positive.

 **3.8. Dividing of the phase plane parts for the predator prey system**

Consider the plane *x*1, *x*2. Of course, the solutions of our equations (numbers of species) have non-negative values. Therefore, it will be sufficient to consider the first quadrant of the plane only.

By equalities (1), each derivative is a product of two values. For both cases, first multiplier consists of two summands. One of them is positive, and the second one is negative. Therefore, this multiplier, in principle, can have the arbitrary sign. However, the second multiplier is non-negative everywhere. Hence, the sign of derivative is the sigh of the corresponding first multiplier.

We conclude, that the value of the derivative  is positive, and the function *x*1 increases at the point *t* if *x*2(*t*)<*ε*1/*γ*1. If that derivative is negative, then the considered function decreases at this point if *x*2(*t*)>*ε*1/*γ*1. Analogically, we determine that the the value of the derivative  is positive, and the function *x*2 increases at the point *t* if *x*1(*t*)>*ε*2/*γ*2. The function *x*2 decreases at this point if *x*1(*t*)<*ε*2/*γ*2. Thus, the phase plane can be divided by four rectangles with fixed directions of change for both functions (see Figure 2):

*x*1<*ε*2/*γ*2, *x*2<*ε*1/*γ*1; *x*1>*ε*2/*γ*2, *x*2<*ε*1/*γ*1; *x*1<*ε*2/*γ*2, *x*2>*ε*1/*γ*1; *x*1>*ε*2/*γ*2, *x*2>*ε*1/*γ*1.



Figure 2. Change directions of the functions *x*1 and *x*2 for the predator–prey model.

**3.9. Evolution of the predator prey system**

Consider the system evolution using of the determined information. Suppose at a time *t* the following inequalities hold

 *x*1(*t*)<*ε*2/*γ*2, *x*2(*t*)<*ε*1/*γ*1. (7)

Then we have the inequalities  and , so the function *x*1 increases and the function *x*2 decreases; see Figure 2.

This property of the system persists all the time until relations (7) are satisfied. These conditions can be violated either in the case when during the increase the value of the function *x*1exceeds the value *ε*2/*γ*2, or when the function *x*2 reaches zero as it decreases. However, the approach of *x*2 to zero is accompanied by the tendency of its derivative to zero in accordance with the second equation (1). Thus, the achievement of the function *x*2 by zero could only be asymptotic. However, with increasing function *x*1 and decreasing *x*2, the derivative not only remains positive, but even grows. Therefore, sooner or later, the value of *x*1 will exceed *ε*2/*γ*2, and the phase curve enters in a region characterized by inequalities

 *x*1(*t*)>*ε*2/*γ*2, *x*2(*t*)<*ε*1/*γ*1. (8)

Under these conditions, the derivatives of both functions are positive, which means that their values will increase with time. This property is observed all the time while the inequalities (8) are true. Obviously, with the growth of the function *x*2, it will ever exceed the value *ε*1/*γ*1, and as a result, the following conditions hold

 *x*1(*t*)>*ε*2/*γ*2, *x*2(*t*)>*ε*1/*γ*1. (9)

In the future, a decrease in the function *u* is observed with a simultaneous increase in *x*2. The situation will change only if one of the relations (9) is violated. Naturally, as the function *x*2 grows, the second of conditions (9) cannot be violated. However, as *x*1 decreases in the end, it becomes less than unity, and we get the inequalities

 *x*1(*t*)<*ε*2/*γ*2, *x*2(*t*)>*ε*1/*γ*1. (10)

This means that both given functions decrease. Considering that as the function *x*1 tends to zero, its derivative tends to zero as well.We conclude that sooner or later the moment of time will come when the function *x*2 becomes less than unity for a positive value of *x*1. Thus, relations (7) will be fulfilled again, which means that the described process is repeated.

It can be prove that the solutions of equations (7) turn out to be periodic functions; see Figure 3. In the phase plane, weobserve closed curves. We have already encountered a similar type of equilibrium (center) in the study of free undamped mechanical and electrical oscillation. At the same time, the trivial equilibrium position (the origin in the phase plane) possess fundamentally different properties. Obviously, only those states that lie on the coordinate axis *v* in the phase plane tend to it. Other phase curves can approach the origin for some time, but subsequently inevitably move away from it, see Figure 3.


Figure 7.8. State functions of the predator – pray model are periodic.

**3.10. Interpretation of results**

Give the interpretation of the results. Suppose that at the initial time, the abundances of prey and predators are small enough. With a small abundance of prey, the abundance of predators is decreasing, and the pray are in a better position due to the small abundance of natural enemies. As a result, their quantity is increasing. The onset of the second stage is due to the fact that a small abundance of predators with an increased abundance of prey will no longer lack food. The abundance of predators begins to grow, and faster and faster due to the still observed increase in the abundance of prey. The third stage begins at that moment in time when the natural increase in the abundance of preyis compensated by their destruction by predators, whose abundanceis steadily growing. With a further increase in the predator population, the abundance of prey begins to gradually decrease, and the velocity of increase in the abundance of predators slows with a decrease in the abundance of prey. In the end, there comes a time when there is already not enough food for the increased abundance of predators, as a result of which their abundancebegins to decline. However, the abundance of prey continues to decrease, as there are still too many predators. The velocity of decrease in prey decreases due to a decrease in the abundance of predators. Then the abundance of prey is reduced to a certain minimum value, after which, due to the continuing decrease in the abundance of predators, the abundance of prey begins to increase. This indicates the onset of a new cycle of the considered process with a return to the initial stage.